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ON THE REFLECTION OF ELASTIC WAVES INCIDENT ON THE BOUNDARY OF A DIATOMIC HALF-SPACE

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Abstract—The paper examines reflection of a plane longitudinal wave incident upon the surface of a half-infinite space the material of which is elastic, and the particles include two atoms of different thermomechanical properties. After establishing the constitutive equations of the constituents, equations of motion are solved by appeal to four quasi-potential functions. Satisfaction of the boundary conditions yields the characteristic equation of the problem which—in contrast to the conventional case, and in agreement with observations—indicates dispersion of waves. An example involving relaxed bonds between the constituents is examined and illustrated by graphs.

1. INTRODUCTION

During the past 30 years a number of workers in mechanics made considerable efforts to introduce into the familiar model of the material continuum some sort of an internal structure known as microstructure.

The attacks in this direction started almost spontaneously, and this has come about in spite of the fact that, for more than two centuries, the concept of the conventional continuum served with distinction in theoretical speculations, and —what was more important —proved to be of value in practical applications in all branches of engineering.

This state of affairs should come as no surprise: throughout the time that the technical people stuck faithfully to the classical concepts, in the hands of physicists the theory of the discrete media made a remarkable progress. This is noticeable most of all in the theory of the solid state of matter in which, apart from many other things, the dynamics of lattices were able to predict theoretically the actual values of some of the elastic moduli[1].

The process of constructing new macroscopic theories of bodies proceeded in several directions. One of the first was the introduction of the polar field theories as modernized versions of the old concept of the couple stress proposed by the brothers Cosserat[2]. In this connection mention should be made of the works of Truesdell and Toupin[3], Toupin[4], Mindlin[5, 6] and Eringen[7, 8] among others.

Another theory, of great generality, was suggested by Green and Rivlin[9, 10]. These authors used the concepts of higher order velocities and deformations plus those of multipolar forces and stresses.

The third line of generalization involved the so-called non-local continua in which the range of interactions between particles is not limited to the conventional infinitesimal neighborhood. The theories of the non-local continua were established for the most part due to the efforts of Green and Rivlin[11], Kroener[12], Eringen[13], Edelen[14] and Kunin[15].

More recently, Demiray[16] made known a study in which he formulated the foundations of a general continuum theory of elastic media with the so-called *diatomic* structure. In this structure the particles are composed of two interacting atoms of different thermomechanical properties and overlapping in their initial positions. A deformation carries them into distinct spatial places with the properties of the bulk material remaining unchanged. The theory is easily extended to *polyatomic* bodies the particles of which comprise an arbitrary number of dissimilar atoms.

In this paper (see also Ref. [17]) we wish to shed some light, however limited, on the characteristic properties of undulatory motion in a diatomic elastic half-infinite space in which a longitudinal wave impinges upon the traction-free surface of the medium.

We first establish the constitutive equations of the medium, and the equations of balance of momenta in terms of four quasi-potential functions.

Fulfillment of the boundary conditions leads to the characteristic equation of the problem which—in contrast with the conventional case and in agreement with the observations—reveals a *dispersion* of waves.† The dispersion equation is analyzed in more detail for a material in which interactions between the constituents are partially relaxed. The analysis is illustrated by graphs.

2. GENERAL EQUATIONS

Let the x_1–x_3 plane of a Cartesian rectangular coordinate system $\{x_i\}$, $i = 1, 2, 3$, coincide with the surface of a half-infinite material medium the x_2-axis of the systems directed toward the interior of the medium (Fig. 1). The medium is assumed to be elastic, homogeneous, isotropic, and of a diatomic structure, each of its constituents being designated for convenience by the index in parentheses ((1) or (2)).

Imagine that ideally the half-space $x_2 < 0$ is a vacuum, that the surface of the medium is free from external forces, and that a plane longitudinal wave, P, impinges upon the plane $x_2 = 0$ in the direction PO, the latter making an angle e_1 with the coordinate axis x_1. On reflection, the wave P generates two separate waves: a longitudinal wave, P', and a transverse one, SV', the latter traveling in the direction OSV' and making the angle e_2' with the x_1-axis. Clearly, by assumption, all of the kinematic and dynamic quantities involved in the process are independent of the x_3-coordinate, and the medium is a state of plane strain.

Within the framework of the theory of diatomic media established by Demiray in Ref. [16], the stress components of interest, τ_{ij}, $i = 1, 2$, satisfy the equations of motion

$$\begin{aligned} &\tau^{(i)}_{11,1} + \tau^{(i)}_{21,2} \pm R_1 - \rho_1 \ddot{u}_1 = 0, \\ &\tau^{(i)}_{12,1} + \tau^{(i)}_{22,2} \pm R_2 - \rho_2 \ddot{u}_2 = 0, \quad (i = 1, 2) \end{aligned} \tag{1}$$

where the plus sign (minus sign) refers to medium (1) (to medium (2)), ρ_1 and ρ_2 are the mass densities of media (1) and (2), respectively, the superposed dot denotes time differentiation, and $R_i = a_0(x_i^{(2)} - x_i^{(1)})$, $i = 1, 2$, are the rates of the momentum transfer between the media. The balance of the moment of momentum implies that (for $i = 1, 2$)

$$\tau^{(1)}_{[ij]} + \tau^{(2)}_{[ij]} = 0$$

where

$$\tau^{(i)}_{[ij]} = \tfrac{1}{2}(\tau^{(i)}_{ij} - \tau^{(i)}_{ji}) \tag{2}$$

is the antisymmetric part of the stress tensors.

Since the displacement components are independent of the x_3-coordinate, we introduce four quasi-potential functions $\phi_1(x_1, x_2, t)$ and $\psi_i(x_1, x_2, t)$ such that

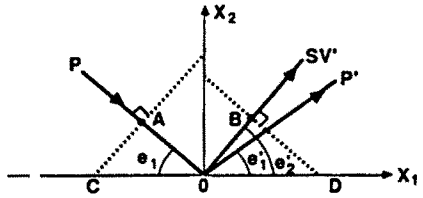

Fig. 1. Geometry of the problem.

† As regards the agreement of the diatomic reflection coefficients with their actual values in diatomic media, too little is known in order to pass an opinion.

$$u_1^{(1)} = \phi_{1,1} + \psi_{1,2}, \quad u_1^{(2)} = \phi_{2,1} + \psi_{2,2}$$
$$u_2^{(1)} = \phi_{1,2} - \psi_{1,1}, \quad u_2^{(2)} = \phi_{2,2} - \psi_{2,1} \tag{3}$$

and the effects of the dilatation and rotation are separated.

The stress components in the case under discussion are

$$\tau_{ij}^{(1)} = 2a_2 e_{ij}^{(1)} + 2a_6 e_{ij}^{(2)} + (a_1 e_{rr}^{(1)} + a_5 e_{rr}^{(2)})\,\delta_{ij} + 2a_7(\omega_{ij}^{(1)} - \omega_{ij}^{(2)}) \tag{4}$$

$$\tau_{ij}^{(2)} = 2a_4 e_{ij}^{(2)} + 2a_6 e_{ij}^{(1)} + (a_3 e_{rr}^{(2)} + a_5 e_{rr}^{(1)})\,\delta_{ij} + 2a_7(\omega_{ij}^{(2)} - \omega_{ij}^{(1)}). \tag{5}$$

Here $i, j = 1, 2$, the a_i's are material coefficients, δ_{ij} is the Kronecker delta

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \quad \omega_{ij} = \tfrac{1}{2}(u_{i,j} - u_{j,i}) \tag{6}$$

u_i, $i = 1, 2$, is the displacement component, a comma denotes the differentiation (e.g. $u_{i,j} \equiv \partial u_i / \partial x_j$), and it is summed over the repeated indices (e.g. $e_{rr} \equiv e_{11} + e_{22}$).

A combination of relations (4)–(6) immediately gives

$$\begin{aligned}
\tau_{11}^{(1)} &= (2a_2 + a_1)u_{1,1}^{(1)} + (2a_6 + a_5)u_{1,1}^{(2)} + a_1 u_{2,2}^{(1)} + a_5 u_{2,2}^{(2)} \\
\tau_{22}^{(1)} &= (2a_2 + a_1)u_{2,2}^{(1)} + (2a_6 + a_5)u_{2,2}^{(2)} + a_1 u_{1,1}^{(1)} + a_5 u_{1,1}^{(2)} \\
\tau_{21}^{(1)} &= (a_2 - a_7)u_{1,2}^{(1)} + (a_2 + a_7)u_{2,1}^{(1)} + (a_6 + a_7)u_{1,2}^{(2)} + (a_6 - a_7)u_{2,1}^{(2)}.
\end{aligned} \tag{7}$$

Thus, by inserting eqns (7) into the equations of motion, eqns (1), one arrives at two independent systems of coupled partial differential equations

$$\begin{aligned}
b_1\nabla^2\phi_1 + b_2\nabla^2\phi_2 + b_0(\phi_2 - \phi_1) &= \rho_1\ddot{\phi}_1 \\
b_2\nabla^2\phi_1 + b_3\nabla^2\phi_2 + b_0(\phi_1 - \phi_2) &= \rho_2\ddot{\phi}_2
\end{aligned} \tag{8}$$

$$\begin{aligned}
b_4\nabla^2\psi_1 + b_5\nabla^2\psi_2 + b_0(\psi_2 - \psi_1) &= \rho_1\ddot{\psi}_1 \\
b_5\nabla^2\psi_1 + b_6\nabla^2\psi_2 + b_0(\psi_1 - \psi_2) &= \rho_2\ddot{\psi}_2
\end{aligned} \tag{9}$$

where $\nabla^2 = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$. The new material coefficients, b_i's, are now defined in terms of the earlier ones as follows:

$$\begin{aligned}
b_1 &= a_1 + 2a_2, \quad b_2 = a_5 + 2a_6, \quad b_3 = a_3 + 2a_4 \\
b_4 &= a_2 - a_7, \quad b_5 = a_6 + a_7, \quad b_6 = a_4 - a_7.
\end{aligned} \tag{10}$$

Guided by the general correspondence principle stating that, after some appropriate simplifications, a diatomic problem should convert into a standard, monoatomic problem, we select the solution of eqns (8) and (9) in the form

wave P, incident

$$\begin{aligned}
\phi_1 &= A\, e^{ik(x_1 - p_1 x_2 - ct)} \\
\phi_2 &= B\, e^{ik(x_1 - p_2 x_2 - ct)}
\end{aligned} \tag{11a}$$

wave P', reflected

$$\begin{aligned}
\phi_1' &= A'\, e^{ik(x_1 + p_1 x_2 - ct)} \\
\phi_2' &= B'\, e^{ik(x_1 + p_1 x_2 - ct)}
\end{aligned} \tag{11b}$$

wave SV', reflected

$$\psi_1' = C' \, e^{ik(x_1+p_2x_2-ct)}$$
$$\psi_2' = D' \, e^{ik(x_1+p_2x_2-ct)}. \tag{11c}$$

Here k, p_1, p_2, and c are some, as yet unspecified, coefficients. The reasons for taking the solution of the problem in question in the just recorded form are threefold.

(1) The presence of the identical exponential multiplier $e^{ik(x_1-ct)}$ in all of eqns (11) guarantees the satisfaction of the imposed boundary conditions at the plane $x_2 = 0$.

(2) Since the coefficients $-p_1$, p_1 and p_2 are equal to the tangents of angles e_1, e_1' and e_2', respectively, the presence of the identical coefficients p's in each of the pairs of eqns (11) makes sure that the pairs of the associated wave constituents propagate in the same direction.

(3) The somewhat hidden assumption of the equality of the angles of the incidence and reflection of the P wave is justified by the very fact that expressions (11) satisfy all of the conditions of the problem.

Ipso facto then, the only difference between the representation of the constituents in each pair of eqns (11) consists in the value of the amplitude coefficients (such as A and B, for example).

We now observe that the equations of the wave fronts of the P, P' and SV' waves are all of the same form $x_1 - sx_2 - ct = \text{const.}$ ($s = -p_1, p_1, p_2$). Consequently, by following exactly the argument of the conventional wave theory we arrive at the generalized Snell's law

$$\frac{\cos e_1}{c_1^{\text{diat.}}} = \frac{\cos e_2'}{c_2^{\text{diat.}}} \tag{12}$$

where $c_1^{\text{diat.}} = c/(1+p_1^2)^{1/2}$ and $c_2^{\text{diat.}} = c/(1+p_2^2)^{1/2}$ are the velocities of the longitudinal and transverse waves in an infinite diatomic medium, respectively. Again, $p_1 = \tan e_1$ and $p_2 = \tan e_2'$, so that $c = c_1^{\text{diat.}}/(\cos e_1) = c_2^{\text{diat.}}/(\cos e_2')$, and in eqns (11a)-(11c) the appropriate wave numbers turn out to be $k_1^* = k/(\cos e_1)$ and $k_2^* = k/(\cos e_2')$, respectively. The corresponding frequencies, ω, and wavelengths, l, thus are $kc_1^{\text{diat.}}/(\cos e_1)$, $kc_2^{\text{diat.}}/(\cos e_2)$ and $(2\pi \cos e_1)/k$, $(2\pi \cos e_2')/k$, respectively. With all this in mind, we substitute expressions (11) into the field equations, eqns (8) and (9), and, after some manipulations, arrive at the conclusion that a non-trivial solution of eqns (8) requires that

$$(1+p_1^2)^2k^4v_1 + (1+p_1^2)k^2v_2 + v_3 = 0 \tag{13a}$$

where†

$$v_1 = b_1b_3 - b_2^2, \quad v_2 = -(\rho_1b_3 + \rho_2b_1)k^2c^2 + b_0(b_1+b_3+2b_2)$$
$$v_3 = (\rho_1\rho_2k^2c^2 - (\rho_1+\rho_2)b_0)k^2c^2. \tag{13b}$$

A similar condition for a non-trivial solution of system (9) is obtained by replacing coefficients b_1, b_2, b_3 and p_1 by the coefficients b_4, b_5, b_6 and p_2, respectively, and introducing some new coefficients v_i' that replace the old ones, v_i, $i = 1, 2, 3$. The operation (it turns out) yields four values for each of the parameters p_1 and p_2 as functions of the wave number and the wave frequency. They are

$$p_{1(1,2,3,4)} = \alpha, -\alpha, \beta, -\beta$$
$$p_{2(1,2,3,4)} = \alpha_1' - \alpha_1' + \beta_1' - \beta_1' \tag{14a}$$

where

† Note that the products k^2c^2 imply frequencies.

$$\alpha = \left[\frac{1}{2v_1k^2}[-v_2+(v_2^2-4v_1v_3)^{1/2}]-1\right]^{1/2}$$

$$\beta = \left[\frac{1}{2v_1k^2}[-v_2-(v_2^2-4v_1v_3)^{1/2}]-1\right]^{1/2} \tag{14b}$$

with the corresponding expressions for p_2 after replacing v_i, $i = 1,2,3$, by v_i', $i = 1,2,3$, respectively.

As far as the relations between the quantities c, $c_1^{\text{diat.}}$, k, and ω are concerned, it is informative to rewrite eqn (13a), and its companion equation, both solved with respect to the frequency ω. This gives

$$\rho_1\rho_2\omega^4-\omega^2[b_0(\rho_1+\rho_2)+(\rho_1b_3+\rho_2b_1)(1+p_1^2)k^2] + b_0(b_1+b_3+2b_2)(1+p_1^2)k^2+(b_1b_3-b_2^2)(1+p_1^2)^2k^4 = 0 \tag{15a}$$

for the longitudinal waves, and

$$\rho_1\rho_2\omega^4-\omega^2[b_0(\rho_1+\rho_2)+(\rho_1b_6+\rho_2b_4)(1+p_2^2)k^2] + b_0(b_4+b_6+2b_5)(1+p_2^2)k^2+(b_4b_6-b_5^2)(1+p_2^2)^2k^4 = 0 \tag{15b}$$

for the transverse waves. For the grazing incidence, that is for $p_1 = p_2 = 0$, the foregoing equations reduce, as they should, to the equations derived directly in Ref. [16] for the propagation of plane waves in an infinite diatomic space. It is good to note that either of eqns (15) signals the existence of the acoustical and optical branches of the undulatory spectrum in question, the coefficients α and α' being associated with the optical and the coefficients β and β' with the acoustical branch, respectively. Again, the coefficients α and β are related to the longitudinal wave, and the coefficients α' and β' involve the transverse waves. As regards eqns (13), by relating wave speed with wave number they clearly exhibit the fact that the wave motions in diatomic media are subject to *dispersion*.

If one wishes to reduce the diatomic results to those of the conventional theory one has simply to set $\rho_1 = \rho_2 = \rho$ and sever the bonds between the constituent materials by assuming that $b_0 = b_2 = b_5 = 0$. Equations (13) then furnish, for example

$$p_1 = \left(\frac{c^2}{c_1^2}-1\right)^{1/2}, \quad p_2 = \left(\frac{c^2}{c_2^2}-1\right)^{1/2} \tag{16}$$

where $c_1^2(=b_1/\rho) \equiv (\lambda+2\mu)/\rho$ and $c_2^2(=b_3/\rho) \equiv \mu/\rho$ are the squares of the classical velocities of the longitudinal and transverse waves, respectively (compare, e.g. eqns (2.7) in Ref. [18]).

Some additional features of the problem at hand are brought out by solving the biquadratic equations, eqns (15a) and (15b), and finding the explicit relations between the wave frequencies and the wave numbers. With regard to the longitudinal waves eqn (15a) then yields

$$\omega^2 = \frac{j_1+j_2k^2 \pm [(j_1+j_2k^2)^2-4\rho_1\rho_2(j_1k^2+j_4k^4)]}{2\rho_1\rho_2} \tag{17a}$$

where the plus (minus) sign denotes the optical (acoustical) branch, and

$$j_1 = b_0(\rho_1+\rho_2), \quad j_2 = (\rho_1b_3+\rho_2b_1)$$
$$j_3 = b_6(b_1+b_3+2b_2), \quad j_4 = b_1b_3-b_2^2. \tag{17b}$$

A similar result is gained for the transverse waves. For long waves, the province of the

classical linear elasticity, there is $k \ll 1$, and by disregarding powers of k higher than the second one, one gets for the acoustical branch of the longitudinal waves

$$\omega^2 = \frac{b_1+b_3+2b_2}{\rho}(1+p_1^2)k^2 \tag{18a}$$

likewise

$$\omega^2 = \frac{b_4+b_6+2b_5}{\rho}(1+p_2^2)k^2 \tag{18b}$$

for the same branch of the transverse waves.

It is worth noting that in the terms $k^2(1+p_1^2)$ and $k^2(1+p_2^2)$ in all of the formulas above one recovers the appropriate wave numbers referred to earlier as k_1^* and k_2^*, respectively.

In the limit case, i.e. for $k \to 0$, the frequency ω vanishes so that the cut-off frequency of the waves becomes equal to zero. Again by setting $b_2 = b_5 = 0$, and identifying b_1 and b_3 as well as b_4 and b_6 with $\lambda+2\mu$ and μ, respectively, one reduces relations (18) to their classical form.

Returning now to the field equations, eqns (8) and (9), we write their solutions in the form

$$\begin{aligned}\phi_1 &= (A_1 e^{-ik\alpha x_2} + A_2 e^{-ik\beta x_2} + A_1' e^{ik\alpha x_2} + A_2' e^{ik\beta x_2})\, e^{ik(x_1 - ct)}\\ \phi_2 &= (B_1 e^{-ik\alpha x_2} + B_2 e^{-ik\beta x_2} + B_1' e^{ik\beta x_2} + B_2' e^{ik\beta x_2})\, e^{ik(x_1 - ct)}\end{aligned} \tag{19}$$

where the first two terms (the last two terms) refer to the incident (reflected) longitudinal waves. Likewise, the representations of the reflected transverse waves are found to be

$$\begin{aligned}\psi_1 &= C_1' e^{ik(x_1 + p_2 x_2 - ct)}\\ \psi_2 &= D_1' e^{ik(x_1 + p_2 x_2 - ct)}\end{aligned} \tag{20}$$

At this stage we have to turn our attention to the stress conditions on the boundary of the half-space. This is done in the next section.

3. BOUNDARY CONDITIONS

As already noted, the surface $x_2 \simeq 0$ of the medium is free from external load. The boundary conditions, therefore, are

$$\tau_{22}^{(1)} = \tau_{22}^{(2)} = 0, \tag{21a}$$

$$\text{at } x_2 = 0$$

$$\tau_{21}^{(1)} = \tau_{21}^{(2)} = 0, \tag{21b}$$

for all values of x_1 and at all times t.

Let us now introduce a partially new notation that extends slightly our earlier notation, eqns (10), namely

$$\begin{aligned} &b_1 = 2a_2 + a_1, \quad b_2 = 2a_6 + a_5, \quad b_3 = 2a_4 + a_3\\ &b_4 = a_2 - a_7, \quad b_5 = a_6 + a_7, \quad b_6 = a_4 - a_7\\ &b_7 = a_2 + a_7, \quad b_8 = a_6 - a_7, \quad b_9 = a_4 + a_7. \end{aligned} \tag{22}$$

We next substitute eqns (16) and (17) into eqns (3) and the result into eqns (7) and their

companion equations. After some algebra we then arrive at the following final form of eqns (21a) and (21b):

for constituent (1)

$$(a_1+b_1p_1^2)(A+A')+(a_5+b_2P_1^2)(B+B')+(a_1-b_1)p_2C'+(a_5-b_2)p_2D'=0 \quad (23)$$

$$(b_4+b_7)p_1(A-A')+(b_5+b_8)p_1(B-B')+(b_7-b_4p_2^2)C'+(b_8-b_5p_2^2)D'=0 \quad (24)$$

for constituent (2)

$$(a_5+b_3p_1^2)(A+A')+(a_3+b_3p_1^2)(B+B')+(a_5-b_2)p_2C'+(a_3-b_3)p_2D'=0 \quad (25)$$

$$(b_5+b_8)p_1(A-A')+(b_6+b_9)p_1(B-B')+(b_8-b_5p_2^2)C'+(b_9-b_6p_2^2)D'=0. \quad (26)$$

The just listed system of four linear algebraic equations includes four pairs of and two individual unknown amplitude coefficients, altogether six unknowns. This being so, the representation of each of the unknown coefficients individually in terms of the remaining ones cannot be achieved, and the only remaining alternative is to find the ratios of the unknowns. The latter, however, turn out to be of a fairly complicated form, and their analysis would rather obscure than clarify the salient points of the diatomic problems. Taking into account this fact, it seems more productive to select a different line of approach; namely, instead of adhering to a full generality, simplify the adopted model of the diatomic medium by relaxing in part the bonds existing between the constituent materials. For mathematical convenience we adopt this procedure in the next section.

4. ILLUSTRATIVE EXAMPLES

An inspection of the constitutive equations, eqns (4) and (5), as well as of the field equations, eqns (8) and (9), reveals the fact that the interactions between the constituent materials are represented by the coefficients a_0, a_5, a_6 and a_7 (see also Table 1 in Ref. [17]). In this connection we examine briefly the following particular cases.

Case 1. First, we relax partially the interatomic connections by setting a_5, a_6 and a_7 equal to zero, but leaving the rates of the momentum transfer, that is, the coefficient a_0, intact.† As a result, the associated coefficients b_2, b_5 and b_8 vanish, and in addition to that one has $b_4=b_7=a_2$, and $b_6=b_9=a_4$, so that eqns (23)–(26) become

$$\begin{aligned}[a_1+(a_1+2a_2)p_1^2](A+A')-2a_2p_2C'=0\\ 2p_1(A-A')+(1-p_2^2)C'=0\end{aligned} \quad (27)$$

$$\begin{aligned}[a_3+(a_3+2a_4)p_1^2](B+B')-2a_4p_2D'=0\\ 2p_1(B-B')+(1-p_2^2)D'=0.\end{aligned} \quad (28)$$

After a simple elimination of constants the system above immediately gives the ratios of the amplitudes

$$\frac{A'}{A}=\frac{4p_1p_2a_2+(1-p_2^2)P_1}{4p_1p_2a_2-(1-p_2^2)P_1} \quad (29a)$$

and

$$\frac{B'}{B}=\frac{4p_1p_2a_4+(1-p_2^2)P_2}{4p_1p_2a_4-(1-p_2^2)P_2} \quad (29b)$$

of the reflected and incident longitudinal waves, as well as

† It is of interest to note that (as mentioned in Ref. [16]) the constant a_0 plays the role of the spring constant assumed in lattice dynamics (p. 143 in Ref. [19]) to act between the atoms of the lattice.

Table 1

	Diatomic coefficients	Monoatomic coefficients
Constituent (1)	a_1	λ_1
	a_2	μ_1
Constituent (2)	a_3	λ_2
	a_4	μ_2

$$\frac{C'}{A} = \frac{4P_1}{4p_1p_2a_2-(1-p_2^2)P_1} \tag{30a}$$

$$\frac{D'}{B} = \frac{4P_2}{4p_1p_2a_4-(1-p_2^2)P_2} \tag{30b}$$

of the transverse reflected and longitudinal incident waves. Here

$$P_1 = a_1+(a_1+2a_2)p_1^2, \quad P_2 = a_3+(a_3+2a_4)p_1^2. \tag{31}$$

Case 2. If to the domain of the diatomic media one extends the concept of the so-called Poisson material (for which in the conventional case there is $\lambda = \mu$), then $a_1 = a_2$ and $a_3 = a_4$, and ratios (29) and (30) become explicitly independent of the material constants, and formally identical with the classical formulas (see, e.g. Refs [20, 21]).†

Case 3. A final simplification is gained if (considering the existing parallelism between the material constants) one sets $c_1^{\text{diat.}} = c_1$ and $c_2^{\text{diat.}} = c_2$, where c_1 and c_2 are the well-known conventional longitudinal and transverse wave speeds. In this case, eqns (29) and (30) become identical with their classical counterparts.

Case 4. With the intent to arrive at some more tangible, numerical results we return to Case 1, and in Table 1 recall the correspondence between the diatomic and the monoatomic elastic coefficients (see also eqns (4) and (5)). In light of these correlations, it seems permissible to assume that a_1 and a_2, on the one hand, and a_3 and a_4, on the other, are connected each with the other by the agency of some hypothetical Poisson's constants, ν_1 and ν_2, respectively,‡ in the same manner as are the classical Lamé constants; thus, $a_1/a_2 = 2\nu_1/(1-2\nu_1)$, for example. By the symmetry of the problem, we then concentrate our attention on constituent (1) alone, and consider two subcases: $\nu_1 = 0$ and 0.25, the latter corresponding to the Poisson type material. In the first of these subcases there is $a_1/a_2 = 0$, in the second one $a_1/a_2 = 1$. Next, we introduce the notation

$$\frac{c_1^2}{c_2^2} \equiv \kappa \equiv \frac{c_{1\,\text{diat.}}^2}{c_{2\,\text{diat.}}^2} \tag{32}$$

so that

$$p_2^2 = \kappa(1+p_1^2)-1 \tag{33}$$

both in the conventional and in the diatomic context. Here the wave velocities $c_{i\,\text{diat.}}$, $i = 1, 2$, are identified with the corresponding velocities of very long waves ($k \ll 1$) traveling in an infinite diatomic medium (see eqns (5.16) in Ref. [16]). For $b_2 = 0$ the former become

† It should be clear that for the Poisson (also known as Poisson–Duhamel, see p. 143 of Ref. [22]) material, the Poisson coefficient $\nu = 1/4$.

‡ The Poisson constant notation should cause no confusion with our previous ν notation in eqns (13).

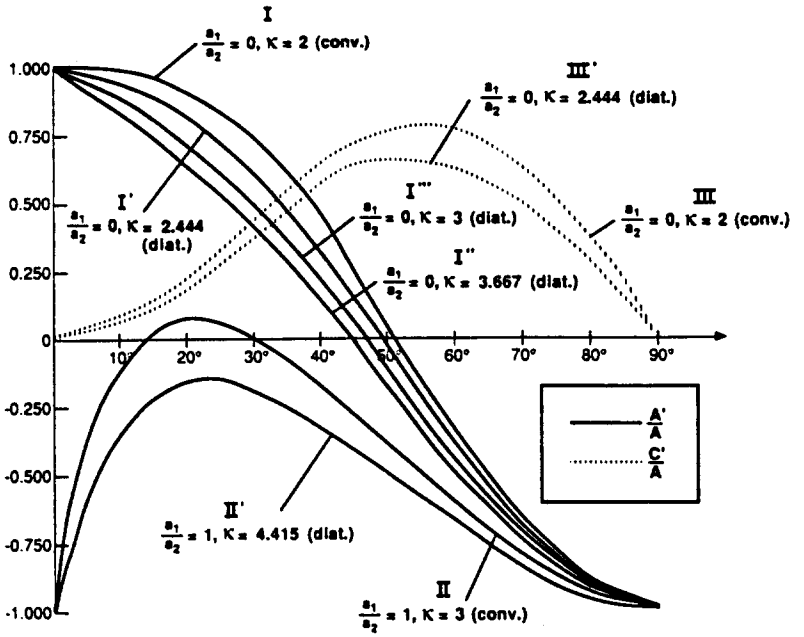

Fig. 2. Ratios of reflection amplitudes vs angle e_1.

$$c^2_{1\,\text{diat.}} = \frac{b_1+b_3}{\rho_1+\rho_2}, \quad c^2_{2\,\text{diat.}} = \frac{b_4+b_6}{\rho_1+\rho_2}. \tag{34}$$

The values of the ratios A'/A and C'/A of the wave amplitudes vs the values of the angle e_1 are exhibited by the graphs in Fig. 2.

(a) The ratios A'/A are depicted by two sets of curves. Curves I and II are calculated according to the conventional theory assuming that $\nu = 0$ and 1/4, respectively. Curves I', I'', I''' and II' are found by appeal to the theory of the diatomic bodies. In particular, curves I'; I''; I'''; II' correspond to the combinations: $\nu_1 = 0$, $\nu_2 = 1/4$; $\nu_1 = 0$, $\nu_2 = 0.4$; $\nu_1 = 0$, $\kappa = 3$; $\nu_1 = 1/4$, $\nu_2 = 0.4$; respectively.

(b) Curves III and III' illustrate the ratios C'/A under the assumption that ν, respectively ν_1, is equal to zero, and that $\kappa = 2$ (the conventional theory), and $\nu_2 = 1/4$, $\kappa = 2.444$ (the diatomic theory), respectively.

5. CONCLUDING REMARKS

The following concluding remarks seem now to be in order.

(1) In contrast to the conclusions of the conventional theory, and in full agreement with the experimental evidence, the undulatory motions in unbounded media in their diatomic aspect turn out to be dispersive.

(2) The interatomic bonds between the constituents of the diatomic media are represented by the interaction coefficients a_5, a_6, a_7 and a_0, the latter corresponding to the spring constant assumed to act between the atoms of the lattices in the lattice dynamics of Born and von Kármán.

(3) The interatomic bonds influence markedly the intensity of the reflected longitudinal and transverse waves, by decreasing the magnitude of their amplitudes. This is so if one compares the diatomic media with their conventional counterparts.

(4) As a simple illustration it is noted that—for $e_1 = 30°$, say—infusion of a constituent with a Poisson's ratio $\nu_2 = 0.4$ into the constituent with a Poisson's ratio $\nu_1 = 0$ (curve I'') decreases the ratio A'/A by more than 40%, as compared with the case of the conventional material the Poisson's ratio ν of which is equal to zero (curve I). Likewise, the diatomic penetration of the constituent for which $\nu_1 = 0$ into the constituent for which $\nu_2 = 0.25$

(curve III') implies the reduction of the ratio C'/A for $e_1 = 60°$, say, by about 20%, as compared with the conventional material for which $\nu = 0$ (curve III).

(5) While in the simplified situations discussed in Section 4 the reflection coefficients (see eqns (29) and (30)) turned out to be independent of the wave frequency, the very existence of the complex dispersion equations, eqns (15), implies that in a more general treatment the end results may become different.

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